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ON THE CLASSIFICATION OF NILPOTENT SINGULARITIES

BY

M. BERTHIER [a], R. MEZIANI [b], P. SAD [c]

[a] *Chargé de recherches au CNRS, IMR, Campus Beaulieu, Université de Rennes 1, 35042 Rennes Cedex, France*

[b] *Université IBN Tofail, Faculté des sciences, Dép. de Math., B.P. 133, Kenitra, Maroc*

[c] *IMPA, Estrada Dona Castorina 110, J. Botanico, 22460-320, Rio De Janeiro, RJ, Brasil*

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1. Introduction and statement of the results

We deal in this paper with the analytic classification of singular holomorphic foliations defined in a neighborhood of the origin of $\mathbf{C}^2$ and having nonzero linear part. In the “semi simple” case this classification which is closely related to the ratio of the eigenvalues of the linear part, is quasi complete. It appears in particular that the moduli are essentially the ones of a holonomy diffeomorphism (see [2,4,6,7,10,11], ...). Our aim here is to study the analytic classification in the “nilpotent case”, i.e., when both eigenvalues are zero.

Let Λ denote the set of germs at $0 \in \mathbf{C}^2$ of holomorphic 1-forms for which the origin is an isolated singularity. Let also $\mathrm{Diff}(\mathbf{C}^k, 0)$ (respectively $\widehat{\mathrm{Diff}}(\mathbf{C}^k, 0)$) be the group of germs at $0 \in \mathbf{C}^k$ of holomorphic (respectively formal) diffeomorphisms which fix the origin.

Two elements ω_1 and ω_2 of Λ are said to be holomorphically (respectively formally) *conjugated* if there exists an element ϕ of $\mathrm{Diff}(\mathbf{C}^2, 0)$ (respectively $\widehat{\mathrm{Diff}}(\mathbf{C}^2, 0)$) such that $\phi^*\omega_1 \wedge \omega_2 = 0$. We will write $\omega_1 \overset{\text{hol}}{\sim} \omega_2$ (respectively $\omega \overset{\text{for}}{\sim} \omega_2$). The *moduli space* ω^{for} of ω is the set

$$\omega^{\text{for}} = \{\omega_1 \in \Lambda;\, \omega_1 \overset{\text{for}}{\sim} \omega\} / \overset{\text{hol}}{\sim} .$$

If ω^{for} is trivial, ω is said to be *rigid*.

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There are related definitions for subgroups of $\operatorname{Diff}(\mathbf{C},0)$: two subgroups H_1 and H_2 are holomorphically (respectively formally) *conjugated* when there exists an element φ of $\operatorname{Diff}(\mathbf{C},0)$ (respectively $\widehat{\operatorname{Diff}}(\mathbf{C},0)$) such that $\varphi^* H_1 = H_2$, where $\varphi^* H_1$ is the set

$$\varphi^* H_1 = \{\varphi^* h = \varphi^{-1} \circ h \circ \varphi,\ h \in H_1\}.$$

We will put $H_1 \overset{\text{hol}}{\sim} H_2$ (respectively $H_1 \overset{\text{for}}{\sim} H_2$). Finally, the *moduli space* H^{for} of a subgroup H of $\operatorname{Diff}(\mathbf{C},0)$ is the set

$$H^{\text{for}} = \{H_1 \textit{ subgroup of } \operatorname{Diff}(\mathbf{C},0),\ H_1 \overset{\text{for}}{\sim} H\}/\overset{\text{hol}}{\sim}.$$

We say that H is *rigid* when H^{for} is trivial. Accordingly, the same notions can be defined for the k-uples $(f_1,\ldots,f_k)$ of elements of $\operatorname{Diff}(\mathbf{C},0)$. We will write $\varphi^*(f_1,\ldots,f_k)$ for $(\varphi^* f_1,\ldots,\varphi^* f_k)$.

The ring of the germs of holomorphic functions at $0 \in \mathbf{C}^n$ will be denoted by $\mathcal{O}_n$, its maximal ideal by $\mathcal{M}_n$ and its formal completion by $\widehat{\mathcal{M}}_n$.

An element of Λ is a *reduced singularity* [8] when its 1-jet is linearly equivalent to one of the types

$$x\,dy - \lambda y\,dx,\ \lambda \neq 0,\ \lambda \notin \mathbf{Q}^+; \tag{1}$$

$$x\,dy. \tag{2}$$

In case (1) there exists two (smooth) separatrices which are tangent to the horizontal and vertical axes. To each of these separatrices is associated a holonomy diffeomorphism which determines the differential equation. More precisely, we have the following (see [9])

THEOREM 1.1. – *Consider two* 1-*forms*

$$\omega_j = x\,dy - \mu y\big(1 + A_j(x,y)\big)\,dx, \quad A_j \in \mathcal{M}_2, \tag{3}$$

of type (1), $j = 1, 2$. *Let* $h_j \in \operatorname{Diff}(\mathbf{C},0)$ *be the holonomy diffeomorphisms of* $\{y = 0\}$ (*associated to* $\omega_j = 0$) *calculated for some transversal section* $\Sigma_0 = \{x = x_0\}$. *Assume that there exists an element* φ *of* $\operatorname{Diff}(\mathbf{C},0)$ *such that* $\varphi^* h_2 = h_1$. *Then there exists* $\Phi \in \operatorname{Diff}(\mathbf{C}^2,0)$ *of the* *form* $\Phi(x,y)=(x,yg(x,y))$ *such that*

$$\Phi^*\omega_2\wedge\omega_1=0,\qquad \Phi_{|\Sigma_0}=\varphi. \tag{4}$$

We will say that such a Φ is *fibered in the variable* $x\in\mathbf{C}$.

In case (2) ("saddle-node singularity"), it is known that there exists a separatrix tangent to the x-axis at $0\in\mathbf{C}^2$; it is called *strong separatrix*. The y-axis may or may not have a separatrix tangent to it at $0\in\mathbf{C}^2$. If this is the case, we call it *center separatrix*. In this paper we deal only with the last situation and we assume that the separatrices are horizontal and vertical axes. We refer to the singularity as being of type $(2)_{\mathrm{c}}$. One of or objectives is to prove

THEOREM 1.2. – *Let* $\omega_j\in\Lambda,\ j=1,2,$

$$\omega_j=x\bigl(1+A_j(x,y)\bigr)\,dy-yB_j(x,y)\,dx \tag{5}$$

of type $(2)_{\mathrm{c}}$. *Let also* $h_j\in\mathrm{Diff}(\mathbf{C},0)$ *denote the holonomy diffeomorphisms of* $\{y=0\}$ (*associated to* $\omega_j=0$) *calculated for some transversal section* $\Sigma_0=\{x=x_0\}$. *Assume there exists an element* φ *of* $\mathrm{Diff}(\mathbf{C},0)$ *such that* $\varphi^*h_2=h_1$. *Then there exists* $\Phi\in\mathrm{Diff}(\mathbf{C}^2,0)$, *fibered in* $x\in\mathbf{C}$, *such that*

$$\Phi^*\omega_2\wedge\omega_1=0,\qquad \Phi_{|\Sigma_0}=\varphi. \tag{6}$$

The fact that the holonomy diffeomorphism associated to the strong separatrix classifies the differential equation has already been proved in [6]. We give here another proof (in the case of type $(2)_{\mathrm{c}}$) more adapted to the study of some types of nilpotent singularities.

A singularity $\Omega=0$, $\Omega\in\Lambda$, is called *nilpotent* if the 1-jet of Ω is linearly equivalent to $y\,dy$. It is well known that such a Ω possesses a normal formal form of the type

$$\Omega^{n,p}=d\bigl(y^2+x^n\bigr)+x^pU(x)\,dy \tag{7}$$

where $n-1\geqslant 2$ is the milnor number, $p\geqslant 2$ is an integer and U is an element of $\mathbf{C}[[x]]$ with $U(0)\neq 0$ (see [12]). In the following, $\Sigma_{n,p}$ denotes the set of thoses Ω in Λ for which there exists a normal formal form of the type $\Omega^{n,p}$.

Although p is not an invariant for the differential equation, the relations $2p>n$, $2p=n$ or $2p<n$ are so. The first two cases were

studied in [3] and [9], were a “generic” rigidity was found. We are interested here in the case $2p < n$, which we assume from now on.

The equation $\Omega = 0$ has the same desingularization as $\Omega^{n,p} = 0$, obtained after p blowing-ups

$$y = x_1 x,\ x_1 = x_2 x, \dots, x_{p-2} = x_{p-1} x,\ x = u x_{p-1}.$$

The exceptional divisor D_p is a linear chain of invariant projective lines $P_j,\ j = 1, \dots, p$ (by the way, since the desingularization process stops after p blowing-ups, we see that p is also an invariant of the equation). All the corners $m_{p-j+1} = P_j \cap P_{j+1}$ are linearizable singularities of type (1), with $\lambda \in \mathbf{Q}^-$; the last projective line P_p contains two other singularities m_{-i}, of type (1), and m_i of type (2) (P_p contains its strong separatrix). We study here the situation where m_i is a singularity of type $(2)_{\mathrm{c}}$. Together with one of the separatrices of m_{-i}, there exist two smooth separatrices transverse to D_p, therefore we can see this is the case before starting the desingularization process. We introduce the notation $\Sigma_{n,p}^c$ for these 1-forms. We fix now a transversal section $\{u = u_0\}$ to P_p, $u_0 \notin \{m_0, m_i, m_{-i}\}$. The holonomy group of $P_p \setminus \{m_0, m_i, m_{-i}\}$ is generated by the holonomy diffeomorphisms h_0 and h_i associated to the singularities m_0 and m_i. It is easy to see that h_0 is a periodic local diffeomorphism (with period p) and h_i is tangent to the identity map.

THEOREM 1.3. – *Let Ω_1 and Ω_2 be elements of $\Sigma_{n,p}^c$, $2p < n$. They are holomorphically conjugated if and only if there exists $\psi \in \mathrm{Diff}(\mathbf{C}, 0)$ such that $\psi^*(h_0^{(1)}) = h_0^{(2)}$ and $\psi^*(h_i^{(1)}) = h_i^{(2)}$.*

This theorem allows us to compare Ω^{for} with $(h_0, h_i)^{\mathrm{for}}$ and find a condition for the rigidity of Ω in $\Sigma_{n,p}^c$.

THEOREM 1.4. – *Let Ω be an element of $\Sigma_{n,p}^c$, $2p < n$; then Ω is rigid in $\Sigma_{n,p}^c$ if and only if its holonomy group is not abelian.*

In Section 2 of this paper we give two different proofs of Theorem 1.2. Section 3 (respectively 4) is devoted to the proof of Theorem 1.3 (respectively 1.4). The last Section contains complements. We show in particular, by constructing an example, that the methods used in this paper are not enough to extend the classification to the singularities in $\Sigma_{n,p} \setminus \Sigma_{n,p}^c$ $(2p < n)$.

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2. Fibered conjugacies in the case $(2)_c$

Let $\omega \in \Lambda$ be a saddle-node singularity of type $(2)_c$; we assume that the separatrices coincide with the horizontal and vertical axes:

$$\omega = x\big(1 + A(x, y)\big)\,dy - yB(x, y)\,dx, \qquad A, B \in \mathcal{M}_2. \tag{8}$$

Proof of Theorem 1.2 using normal formal forms. – It is known [6] that ω is formally conjugated to a normal form

$$\omega_{k,\lambda} = x(1 + \lambda y^k)\,dy - y^{k+1}\,dx \tag{9}$$

for some $k \in \mathbf{N}^*$ (the *Milnor number* of the singularity is $k + 1$) and $\lambda \in \mathbf{C}$ (the other *formal invariant* of the saddle-node). Furthermore, the conjugacy is given by

$$\widehat{\Phi}_0(x, y) = \left(x\left(1 + \sum_{n\geqslant 0} a_n(x)y^n\right), y\sum_{n\geqslant 0} b_n(x)y^n\right), \tag{10}$$

where the functions a_n and b_n are holomorphic functions defined in the same disc with center at $0 \in \mathbf{C}$ (independant of n). $\square$

LEMMA 2.1. – *There exists* $\widehat{\Phi} \in \widehat{\mathrm{Diff}}(\mathbf{C}^2, 0)$ *such that*

$$\widehat{\Phi}(x, y) = \left(x, y\sum_{n\geqslant 0} f_n(x)y^n\right), \qquad \widehat{\Psi}^*\omega \wedge \omega_{k,\lambda} = 0, \tag{11}$$

where the functions f_n *are holomorphic, defined in the same disc with center at* $0 \in \mathbf{C}$.

Proof. – We define $\widehat{\Phi}_1 \in \widehat{\mathrm{Diff}}(\mathbf{C}^2, 0)$ in order to have $\widehat{\Phi}_1 \circ \widehat{\Phi}_0$ satisfying the lemma. Let us take

$$X = x\frac{\partial}{\partial x} + \frac{yB(x, y)}{1 + A(x, y)}\frac{\partial}{\partial y}. \tag{12}$$

It has associated a flow

$$X_t(x,y)=\left(e^t x,\, yU(t,x,y)\right),\quad U\in\mathcal{O}_3,\ U(t,0,0)\neq 0.$$

For any function $t(x,y)\in\mathcal{M}_2$ we have that $\Psi(x,y)=X_{t(x,y)}(x,y)$ satisfies $\Psi^*\omega\wedge\omega=0$. This property remains true when we consider an element $\hat{t}(x,y)$ of $\widehat{\mathcal{M}}_2$ because we may proceed to computations using series developments. We take $\hat{t}'(x,y)\in\widehat{\mathcal{M}}_2$ in order to have

$$e^{\hat{t}'(x,y)}=1+\sum_{n\geqslant 0}a_n(x)y^n. \tag{13}$$

In particular, $\hat{t}'(x,y)=\sum_{n\geqslant 1}c_n(x)y^n$, where all the functions c_n are holomorphic in the same disc. Finally we put

$$\begin{aligned}\hat{t}(x,y)&=-\hat{t}'\left(\widehat{\Phi}_0^{-1}(x,y)\right),\\ \widehat{\Phi}_1(x,y)&=X_{\hat{t}(x,y)}(x,y),\\ \widehat{\Phi}&=\widehat{\Phi}_1\circ\widehat{\Phi}_0.\end{aligned}\tag{14}$$

It is easy to see that $\widehat{\Phi}$ satisfies the requirements. □

We need now to state a special property of the normal form $\omega_{k,\lambda}$. A simple computation shows that the holonomy diffeomorphism of the strong separatrix is

$$h=\exp(2i\pi X_{k,\lambda})$$

no matter what transversal $\Sigma=\{x=x_0\}$ we choose. Here $X_{k,\lambda}$ is the vector field

$$X_{k,\lambda}=\frac{y^{k+1}}{1+\lambda y^k}\frac{\partial}{\partial y}. \tag{16}$$

LEMMA 2.2. – *Any $\psi\in\widehat{\mathrm{Diff}}(\mathbf{C},0)$ such that $\psi^*h=h$ is convergent and $\Psi^*\omega_{k,\lambda}\wedge\omega_{k,\lambda}=0$ where $\Psi(x,y)=(x,\psi(y))$.*

Proof. – Since $\psi\circ h=h\circ\psi$, it follows (see [3]) that
(i) $\psi_* X_{k,\lambda}=X_{k,\lambda}$,
(ii) $\psi=e^{2i\pi l/k}\exp(t_0 X_{k,\lambda})$ for some $l\in\mathbf{Z}$, $t_0\in\mathbf{C}$.

From (ii) we see that ψ converges. Now the vector field

$$x\frac{\partial}{\partial x}+\frac{y^{k+1}}{1+\lambda y^k}\frac{\partial}{\partial y}$$

is tangent to the foliation given by $\omega_{k,\lambda}=0$, so that (i) implies that $\Psi^*\omega_{k,\lambda}\wedge\omega_{k,\lambda}=0$. $\square$

We remark that Ψ is the extension of ψ to vertical sections obtained by lifting the radial paths in $\{y=0\}$ along the leaves of $\omega_{k,\lambda}=0$ (at least in some neighborhood of $\{y=0\}$).

Now we prove Theorem 1.2. Let ω_j, $j=1,2$, be as in the statement. Since h_1 and h_2 are conjugated, ω_1 and ω_2 have the same normal form $\omega_{k,\lambda}$. Lemma 2.1 gives two diffeomorphisms $\widehat{\Phi}_j\in\widehat{\mathrm{Diff}}(\mathbf{C}^2,0)$,

$$\widehat{\Phi}_j(x,y)=\Big(x,\sum_{n\geqslant 0}f_{n,j}(x)y^n\Big)$$

that satisfy

$$\widehat{\Phi}_j^*\omega_j\wedge\omega_{k,\lambda}=0 \tag{17}$$

all functions $f_{n,j}$ being holomorphic for $|x|\leqslant r$. We extend the diffeomorphism φ, defined in the section Σ_0, to nearby sections by lifting the radial paths in $\{y=0\}$ along the leaves of $\omega_1=0$ and $\omega_2=0$, and get $\Phi(x,y)=(x,\varphi_x(y))$ holomorphic for $0<r_1\leqslant|x|\leqslant r$, $|y|\leqslant\varepsilon$.

Let $\psi=(\hat{\phi}_{2|\Sigma_0})^{-1}\circ\varphi\circ(\hat{\phi}_{1|\Sigma_0})\in\mathrm{Diff}(\mathbf{C},0)$ (by Lemma 2.2). The analogous method of lifting radial paths in $\{y=0\}$ allows us to extend ψ to nearby sections. This extension is Ψ given by Lemma 2.2 but it is also $\widehat{\Phi}_2^{-1}\circ\Phi\circ\widehat{\Phi}_1$. It follows that

$$\Phi(x,y)=\Big(x,\sum_{n\geqslant 1}f_n(x)y^n\Big)$$

where all the functions f_n are holomorphic in the disc $|x|<r$.

We claim now that Φ is in fact holomorphic in a neighborhood of $0\in\mathbf{C}^2$. Let $M=\max\{|\Phi(x,y)|,\ r_1\leqslant|x|\leqslant r,\ |y|\leqslant\varepsilon\}$. Cauchy inequalities imply

$$\big|f_n(x)\big|\leqslant\frac{M}{\varepsilon^n},\quad r_1\leqslant|x|\leqslant r. \tag{18}$$

It follows from the maximum principle that

$$\left|f_n(x)\right| \leqslant \frac{M}{\varepsilon^n} \quad \text{for } |x| \leqslant r. \tag{19}$$

We consider $0 < \varepsilon_1 < \varepsilon$. We have the estimates

$$\sum_{n\geqslant 0} \left|f_n(x)y^n\right| \leqslant \sum_{n\geqslant 0} \frac{M}{\varepsilon^n}\varepsilon_1^n < \infty \tag{20}$$

so that $\sum_{n\geqslant 0} f_n(x)y^n$ converges for $|x| \leqslant r$, $|y| \leqslant \varepsilon_1$. This ends the proof. □

Dynamical proof of Theorem 1.2. – The rest of this paragraph is devoted to presenting another proof of Theorem 1.2 which have a dynamic flavor.

We start with two saddle-node singularities with the same Milnor number $k+1 \in \mathbf{N}$ as in (8). Instead of using the normal forms in (9), we will use Dulac's normal form

$$\omega_D = x\bigl(1 + yC(x,y)\bigr)\,dy - y^{k+1}\,dx, \quad C \in \mathcal{O}_2. \tag{21}$$

Dulac's normal form is holomorphically conjugated to the original form (see [6] and [8]). A method analogous to the one used in proving Lemma 2.1 gives a fibered conjugacy in the variable $x \in \mathbf{C}$. We need now to state some facts about the differential equation $\omega_D = 0$.

Let C be holomorphic for $|x| \leqslant R$, $|y| \leqslant R$, $R > 0$ small. For $0 < |a| < R$, we put

$$\Sigma_a = \bigl\{(a,y) \in \mathbf{C}^2,\ |y| \leqslant R\bigr\}.$$

Given a pair Σ_a, $\Sigma_{e^{i\theta}a}$ we may define $h_{a,\theta}(y) \in \Sigma_{e^{i\theta}a}$ as the end point of the path obtained by lifting $t \mapsto (e^{it}a, 0)$, $0 \leqslant t \leqslant \theta$, to the leaf passing through $(a,y) \in \Sigma_a$.

LEMMA 2.3. – *If $R > 0$ is small enough, there exists $r > 0$ such that $h_{a,\theta}$ is defined for $|y| < r$, independently of $a \in \mathbf{C}$, $0 < |a| < R$.*

Proof. – Let $x(t) = ae^{it}$. The lifting of $t \mapsto (ae^{it}, 0)$ satisfies

$$\frac{dy}{dt} = \frac{iy^{k+1}}{1 + yC(x,y)} \tag{22}$$

so that for $R>0$ small enough

$$\left|\frac{dy}{dt}\right| \leqslant \alpha |y|^{k+1} \quad \text{for some } \alpha > 0. \tag{23}$$

It follows that $|\frac{d}{dt}(1/y^k(t))| \leqslant k\alpha$, which implies

$$\left|\frac{1}{y^k(t)} - \frac{1}{y^k(0)}\right| \leqslant k\alpha t, \tag{24}$$

therefore

$$\bigl|y(t)\bigr| \leqslant \frac{|y(0)|}{(1-k\alpha t|y^k(0)|)^{1/k}}. \tag{25}$$

The lemma follows from observing that the estimate in (25) does not depend on a for $0<|a|\leqslant R$. $\square$

COROLLARY 2.4. – *The holonomy diffeomorphism $h_a = h_{a,2\pi}$ is defined for $|y|\leqslant r$, independently of $0<|a|\leqslant R$. The family $\{h_a\}_a$ is uniformly bounded for $|y|\leqslant r$.*

In particular, the holonomy map $(x,y)\mapsto (x,h_x(y))$, defined for $0<|x|\leqslant R$, $|y|\leqslant r$, extends holomorphically to $x=0$, $|y|\leqslant r$ (by Riemann extension theorem) as a diffeomorphism h_0 from $\{x=0,\ |y|\leqslant r\}$ into itself (Hurwicz's theorem).

From (22) we deduce that $h_a(y) = y + 2i\pi y^{p+1} + \cdots$, so that $h_0(y) = y + 2i\pi y^{p+1} + \cdots$. The family $\{h_a(y),\ |y|\leqslant r\}_a$ can be thought as a small deformation of h_0 in the disc $|y|\leqslant r$. The following lemma is a consequence of the description of the dynamics of h_0 in the petals around $0\in \mathbf{C}$ (see [1]).

LEMMA 2.5. – *Let $|a|\in\mathbf{R}$ be small. For each $0<r_1<r/2$, we may associate an integer $n(r_1)$ such that*:

(i) *given $|y|\leqslant r/2$, there exists $n(y,r_1)\in\mathbf{Z}$ such that $|n(y,r_1)|\leqslant n(r_1)$ implies $|h_a^{n(y,r_1)}(y)|\leqslant r_1$,*

(ii) *$|h_a^m(y)|\leqslant r$ for all $|y|\leqslant r_1$, $|m|\leqslant n(r_1)$.*

We consider now the situation given in the statement of Theorem 1.2 for the 1-forms

$$\omega_j = x\bigl(1+yC_j(x,y)\bigr)\,dy - y^{k+1}\,dx, \quad C_j\in\mathcal{O}_2. \tag{26}$$

We have two families of holonomy diffeomorphisms $\{h_a^{(1)}(y),\ |y| \leqslant r\}_a$ and $\{h_a^{(2)}(y),\ |y| \leqslant r\}_a$ associated to $y = 0$, $h_{x_0}^{(1)} = h_1$ and $h_{x_0}^{(2)} = h_2$. Let $f \in \mathrm{Diff}(\mathbf{C}, 0)$ be defined for $|y| \leqslant r' < r,\ f(0) = 0,\ |f'(0)| \leqslant 1$. Again, from the description of the dynamics of $h_0^{(1)}$ and $h_0^{(2)}$ we get

LEMMA 2.6. – *Let $|a|$ be small. Let also, for $r_1 > 0$ close to 0, $n(r_1)$, $n(y, r_1)$ be associated to $h_a^{(1)}$, according to Lemma 2.5. There exists $N \in \mathbf{N}$ such that:*

(i) *given $|y| \leqslant r/2$, $(h_1^{(2)})^{\tilde{n}(y,r_1)}(y)$ belongs to $f(B(0, r_1))$, where $\tilde{n}(y, r_1) = n(y, r_1) + \mathrm{sgn}(n(y, r_1))N$,*

(ii) *$|(h_1^{(2)})^m(y)| \leqslant r$ for all $y \in f(B(0, r_1))$, $|m| \leqslant n(r_1) + N$,*

(iii) *both (i) and (ii) hold true for a diffeomorphism $\tilde{f}$, $\tilde{f}(0) = 0$, defined for $|y| \leqslant r'$, close enough to f.*

The number $N \in \mathbf{N}$ has to be introduce essentially to account for the fact that $|f'(0)| < 1$ possibly.

We are ready to present the proof. We start with the conjugacy φ between h_1 and h_2. Using the real 1-dimensional foliations induced in the solid torus $|x| = |x_0|$ by $\omega_1 = 0$ and $\omega_2 = 0$ we may extend φ to a conjugacy $\varphi_{e^{i\theta}x_0}$ between $h_{e^{i\theta}x_0}^{(1)}$ and $h_{e^{i\theta}x_0}^{(2)}$, defined in $\Sigma_{e^{i\theta}x_0}$ for $|y| \leqslant r,\ r > 0$ independant of $0 \leqslant \theta \leqslant 2\pi$. We want to conjugate $h_{e^{-s}a}^{(1)}$ and $h_{e^{-s}a}^{(2)}$ for $s > 0$ and $|a| = |x_0|$. The foliations defined by $\omega_1 = 0$ and $\omega_2 = 0$ are transversal, for any $|a| = |x_0|$, to $\{x = e^{-t}a,\ 0 \leqslant t < \infty\} \times \mathbf{C}$, so that we may define real 1-dimensional foliations by intersection. We observe that $\{(e^{-t}a, 0),\ 0 \leqslant t < \infty\}$ is a leaf of both foliations. We may then transport φ_2, $|a| = |x_0|$, along these foliations to a map $\varphi_{e^{-s}a}$ which is defined in $\Sigma_{e^{-s}a}$ for small $|y|$ (depending on $s \in \mathbf{R}$). Lemma 2.6 can be applied to extend $\varphi_{e^{-s}a}$ to $|y| \leqslant r/2$ by iteration of $h_{e^{-s}a}^{(1)}$ and $h_{e^{-s}a}^{(2)}$ in the $2k$-petals around the origin. For each $s > 0$, the extension agrees in points which belong to different petals, because $\varphi_{e^{-s}a}$ is defined already in a small neighborhood of the origin and outside it by analytic continuation. Also, given $s_0 > 0$, $0 \leqslant \theta_0 \leqslant 2\pi$ the construction can be done in a uniform way for $|s - s_0|$ small (just apply (iii) of Lemma 2.6 to $\tilde{f} = \varphi_{e^{-s}a}$). It results that the map $(x, y) \mapsto (x, \varphi_x(y))$ is holomorphic in $x = e^{-s}a$, $|y| \leqslant r/2$ for $|s - s_0|$ small and all $|a| = |x_0|$. Consequently it is holomorphic for $0 < |x| \leqslant |x_0|$, $|y| \leqslant r/2$. Riemann's extension theorem garantees the holomorphic extension to $\{x = 0\}$. $\square$

3. Applications to nilpotent singularities

Theorem 1.2 comes in to allow us to employ techniques already used in [9].

Step 1 of the proof of Theorem 1.3. – We have seen in Section 1 that the desingularization scheme of a 1-form $\Omega \in \Sigma^{c}_{n,p}$ consists of a linear chain of invariant projective lines $P_1, \ldots, P_p$ obtained after p blowing-ups

$$y = x_1 x,\ x_1 = x_2 x, \ldots, x_{p-2} = x_{p-1} x,\ x = u x_{p-1}. \tag{27}$$

In the last divisor we find three singularities m_0, m_i, m_{-i}. The singularity $m_0 = P_p \cap P_{p-1}$ is of type (1) with holomorphic first integral, m_{-i} is of type (1) with $\lambda \in \mathbf{Q}^-$ and finally m_i is of type $(2)_c$ with its strong separatrix contained in the divisor. We have then two separatrices transverse to the divisor P_p. We consider coordinates such as to have these separatrices written as $y^2 + x^{2p} = 0$. In the coordinates (u, x) of P_p they will appear as the straight lines $u = \pm i$. Therefore the curves $y^2 + x^{2p} = 0$ are invariant simultaneously for the equations $\Omega = 0$ and $\Omega_0 = py\,dx - x\,dy = 0$. It follows that

$$\Omega \wedge d\left(y^2 + x^{2p}\right) = f_1(x, y)\left(y^2 + x^{2p}\right) dx \wedge dy, \quad f_1 \in \mathcal{O}_2,$$

$$\Omega \wedge (py\,dx - x\,dy) = f_2(x, y)\left(y^2 + x^{2p}\right) dx \wedge dy, \quad f_2 \in \mathcal{O}_2, \tag{28}$$

so that

$$\Omega \wedge \left[f_2 d\left(y^2 + x^{2p}\right) - f_1(py\,dx - x\,dy)\right] = 0 \tag{29}$$

and Ω is conjugated to the 1-form

$$d\left(y^2 + x^{2p}\right) + f(x, y)(py\,dx - x\,dy), \quad f \in \mathcal{M}_2. \tag{30}$$

In fact an easy computation shows that

$$\begin{gathered} f(x, y) = 2i x^{p-1}\big(1 + l(x)\big) + y g(x, y), \\ l \in \mathcal{M}_1,\ g \in \mathcal{O}_2,\ \text{with } lg \not\equiv 0. \end{gathered} \tag{31}$$

Let us denote by $\widetilde{\mathcal{F}}$ the foliation obtained from $\Omega = 0$ after the desingularization (27). Since the leaves of $\Omega = 0$ are transverse to the leaves of $\Omega_0 = 0$ outside $y^2 + x^{2p} = 0$, we have that $\widetilde{\mathcal{F}}$ is transverse to the “Hopf fibration” $\widetilde{\mathcal{F}}_0$ of P_p (which is obtained after applying the transformations (27) to $\Omega_0 = 0$) outside the separatrices $\{u = i\}$, $\{u = -i\}$ and $P_1, \ldots, P_{p-1}$. In the system of coordinates (x, v) where $v = u^{-1}$, $\widetilde{\mathcal{F}}_0$ is given by $dv = 0$.

We remark that the indices of $\widetilde{\mathcal{F}}$ relatively to P_p at the singularities m_0, m_i and m_{-i} are $\mu_0 = -(1 - 1/p)$, $\mu_i = 0$ and $\mu_{-i} = -1/p$, respectively. The holonomy group $\mathbf{H}_\Omega$ of $P_p \setminus \{m_0, m_i, m_{-i}\}$ calculated for the transversal $v = v_0$ is generated by the local holonomy diffeomorphisms h_0 and h_i associated to the singularities m_0 and m_i: $\mathbf{H}_\Omega = \langle h_0, h_i \rangle$. Let us recall that the diffeomorphism h_0 is periodic of period p and that h_i is tangent to the identity map.

Step 2 of the proof. – In order to prove Theorem 1.3, we apply the considerations of step 1 to the equations $\Omega_1 = 0$ and $\Omega_2 = 0$. Clearly, a conjugacy between the 1-forms induces a conjugacy between $h_0^{(1)}$ (respectively $h_i^{(1)}$) and $h_0^{(2)}$ (respectively $h_i^{(2)}$). Conversly, we have $\psi \in \mathrm{Diff}(\mathbf{C}, 0)$ such that $\psi^*(h_0^{(1)}) = h_0^{(2)}$ and $\psi^*(h_i^{(1)}) = h_i^{(2)}$. To show that $\widetilde{\mathcal{F}}^{(1)}$ and $\widetilde{\mathcal{F}}^{(2)}$ are conjugated, we proceed as in [9]. We start by extending ψ to $\widetilde{\Phi}$ in a neighborhood of P_p, except for small balls B_0, B_i and B_{-i} around the singularities m_0, m_i and m_{-i}. This is done in the usual way by lifting paths in P_p along the leaves of the foliations. In order to extend $\widetilde{\Phi}$ to B_0 and B_{-i}, we use Theorem 1.1 stated in Section 1 and to extend it to B_i we use Theorem 1.2. We get then a holomorphic conjugacy between $\widetilde{\mathcal{F}}^{(1)}$ and $\widetilde{\mathcal{F}}^{(2)}$ in a neighborhood of P_p. We continue to extend $\widetilde{\Phi}$ to a neighborhood of the exceptional divisor exactly as it was done in [9]. $\square$

Remark. – There exists a formal version of Theorem 1.3 replacing holomorphic by formal conjugacies.

4. Proof of the rigidity theorem

Let Ω be in $\Sigma^c_{n,p}$. Recall that the group $\mathbf{H}_\Omega$ of $P_p \setminus \{m_0, m_i, m_{-i}\}$ is generated by the diffeomorphisms h_0 and h_i where

– h_0 is a periodic germ of diffeomorphism, $h_0^p = Id$ ($h_0'(0) = e^{2i\pi/p}$),
– h_i is the “strong” holonomy of a saddle node singularity having an analytic center separatrix ($h_i'(0) = 1$).

We would like to compare Ω^{for} with $(h_0, h_i)^{\text{for}}$ but we need to impose a restriction. Consider the set

$$(h_0, h_i)_*^{\text{for}} = \left\{(\tilde{h}_0, \tilde{h}_i) \in \mathrm{Diff}(\mathbf{C}, 0) \times \mathrm{Diff}_{\mathrm{CS}}(\mathbf{C}, 0),\ (\tilde{h}_0, \tilde{h}_i) \overset{\text{for}}{\sim} (h_0, h_i)\right\} / \overset{\text{hol}}{\sim}$$

where $\mathrm{Diff}_{\mathrm{CS}}(\mathbf{C}, 0)$ is the set of germs of diffeomorphisms that can be realised as the "strong" holonomy of some saddle-node singularity having an analytic center separatrix. Let us also denote

$$\Omega_*^{\text{for}} = \left\{\Omega_1 \in \Sigma_{n,p}^c;\ \Omega_1 \overset{\text{for}}{\sim} \Omega\right\} / \overset{\text{hol}}{\sim}$$

the moduli space of Ω "relatively to $\Sigma_{n,p}^c$". As a corollary of Theorem 1.3, we obtain

COROLLARY 4.1. – *Let Ω be in $\Sigma_{n,p}^c$, then $\Omega_*^{\text{for}} = (h_0, h_i)_*^{\text{for}}$.*

Proof. – The proof is the same as in the case $n = 2p$ (see [9]). It is based upon Theorem 1.3 and the synthesis result of [5]. □

In order to prove Theorem 1.4, we need some elementary definitions and properties concerning the analytic classification of holomorphic diffeomorphisms tangent to the identity map. We briefly recall them (cf. [7]).

"Moduli space" and orbits space. We first describe how to obtain the orbits space of the diffeomorphism $\exp 2i\pi X_{m,\lambda}$. Let $\mathcal{U}$ be the covering of the disc $D_\varepsilon = \{|x| < \varepsilon\}$ by the sectors V_j, $j = 0, 1, \ldots, 2m - 1$, defined by

$$V_j = \left\{|x| < \varepsilon,\ \theta_j - \left(\frac{\pi}{m} - \delta\right) < \mathrm{Arg}(x) < \theta_j + \left(\frac{\pi}{m} - \delta\right)\right\},$$

where $\theta_j = -\pi/2m + j\pi/m$ and δ is a small positive real number.

The determinations $F_{m,\lambda}^j$ of the invariant function

$$F_{m,\lambda} = x^{-\lambda} \exp\left(\frac{1}{mx^m}\right)$$

on V_j identify the quotient space $V_j / \exp 2i\pi X_{m,\lambda}$ with a Riemann sphere denoted S_j. The orbits space of $\exp 2i\pi X_{m,\lambda}$ is then obtained by gluing these spheres together according to the following rule: a neighborhood of $\infty \in S_0$ is glued to a neighborhood of $\infty \in S_1$ by the identity map, a neighborhood of $0 \in S_1$ is glued to a neighborhood of $0 \in S_2$ by the identity map etc., ..., and finally a neighborhood of $0 \in S_{2m-1}$ is glued to a neighborhood of $0 \in S_0$ by the diffeomorphism $x \mapsto \exp(-2i\pi\lambda)x$, the monodromy map of $F_{m,\lambda}$ around the origin.

In order to describe the orbits space of a diffeomorphism h formally conjugated to $\exp 2i\pi X_{m,\lambda}$ we replace all identity maps in the previous construction by m pairs

$$\varphi_j = \left(\varphi_j^0, \varphi_j^\infty\right) \in \mathrm{Diff}_1(S_{2j}; 0, \infty), \quad j = 0, 1, \dots, m-1,$$

of local diffeomorphisms of the sphere S_{2j} in $(0, \infty)$ tangent to identity in each of theses points (notice that the last gluing is given by $\exp(-2i\pi\lambda) \circ \varphi_0^0$). The analytic class of h is then determined by the "caracteristic cocycles" φ_j defined modulo conjugacy by a global diffeomorphism of the sphere, i.e., modulo an action of $\mathbf{C}^*$.

Let $\underline{G}^\infty_{m,\lambda}$ denotes the sheaf of sectoriels diffeomorphisms infinitely tangent to identity commuting with $\exp X_{m,\lambda}$. We obtain the "sectorial cocycles" of the "moduli" space

$$H^1\left(S^1, \underline{G}^\infty_{m,\lambda}\right)$$

in the following way. If φ_j is one of the caracteristic cocycle, we define ψ_j^- on $V_{2j-1} \cap V_{2j}$ by

$$F^{2j}_{m,\lambda} \circ \psi_j^- = \varphi_j^0 \circ F^{2j}_{m,\lambda} \tag{32}$$

and ψ_j^+ on $V_{2j} \cap V_{2j+1}$ by

$$F^{2j}_{m,\lambda} \circ \psi_j^+ = \varphi_j^\infty \circ F^{2j}_{m,\lambda}. \tag{33}$$

We have the following result (cf. [7, Chapter II, §6.3], [6, Chapter V, §2.1], [6, Chapter III, Theorem 4.4]).

THEOREM 4.2. – *A germ h of diffeomorphism which is tangent to identity is the holonomy of a saddle node singularity if and only if the elements φ_j^∞ of its caracteristic cocycles are homographies. Moreover,* *this saddle node singularity possesses an analytic center manifold if and only if theses homographies are identity.*

Proof of Theorem 1.4. – According to [3], there are only two cases to deal with.

(1) $\mathbf{H}_\Omega$ is abelian. The pair (h_0, h_i) is formally conjugated to a pair $(\omega x, \exp X_{pk,\lambda})$ where $\omega = \exp(2i\pi/p)$, λ is a complex number and k is an integer. Since the formal conjugacy between $\mathbf{H}_\Omega$ and the group $(\omega x, \exp X_{pk,\lambda})$ transforms ωx in a convergent diffeomorphism, the sectorial cocycles must be equivariant, i.e.,

$$\omega\psi_j^+ = \psi_{j+k}^+\omega \quad \text{and} \quad \omega\psi_j^- = \psi_{j+k}^-\omega \tag{34}$$

(this is because ωx sends the line $\operatorname{Arg}(x) = \theta_i$ on the line $\operatorname{Arg}(x) = \theta_{i+2k}$). The analytic class of $\mathbf{H}_\Omega$ is then determined by the first k caracteristic cocycles of h_i, namely $\varphi_0, \varphi_1, \ldots, \varphi_{k-1}$. In view of the previous Theorem 4.2 they reduce to the elements $\varphi_0^0, \varphi_1^0, \ldots, \varphi_{k-1}^0$. We have finally

$$(h_0, h_i)_*^{\text{for}} = \left[\operatorname{Diff}_1(\mathbf{C}, 0)\right]^k/\mathbf{C}^*.$$

(2) $\mathbf{H}_\Omega$ is exceptionnal and by definition the pair (h_0, h_i) is formally conjugated to the pair $(\omega x, \exp X_{k,0})$ where $\omega = \exp(2i\pi/p)$ satisfies $\omega^k = -1$. We write $\omega = \exp(i\pi(2l+1)/k)$ with l in $\mathbf{Z}$. In this case the equivariance tells us that

$$\omega\psi_j^- = \psi_{j+l}^+\omega \quad \text{and} \quad \omega\psi_j^+ = \psi_{j+l+1}^-\omega \tag{35}$$

(because $\operatorname{Arg}(x) = \theta_i$ is sent on $\operatorname{Arg}(x) = \theta_{i+2l+1}$). The analytic class of $\mathbf{H}_\Omega$ is given by the elements $\varphi_0^\infty, \varphi_1^\infty, \ldots, \varphi_{k-1}^\infty$ of the caracteristic cocycles of h_i. Because of the previous theorem they are trivial and Ω is rigid. □

5. Complements

First, let us state a very simple corollary of Theorem 1.2. We select a special class $\mathcal{C}$ of singularities as follows. Take $\omega \in \Lambda$ such that its singularity has multiplicity m, $m \geqslant 2$. Suppose that

(i) the foliation defined by $\omega = 0$ needs one blowing-up to be desingularized;

(ii) there are exactly $m+1$ smooth transverse separatrices through the origin of $\mathbf{C}^2$ and they are holomorphically equivalent to a family of $m+1$ transverse straight lines.

A singularity of class $\mathcal{C}$ is not dicritical and if it appears a saddle-node singularity in the desingularization, it is necessarily of type $(2)_{\mathrm{c}}$ with its strong separatrix contained in the divisor. Let $\widetilde{\mathcal{F}}$ be the blowing-up of the foliation defined by $\omega = 0$, it possesses $m+1$ singularities $p_1, \ldots, p_{m+1}$ along the divisor. We denote by h_j the holonomy diffeomorphism associated to p_j calculated on some common section and by i_j the index of the singularity p_j relatively to the divisor.

COROLLARY 5.1. – *Let ω_j, $j = 1, 2$, be elements of Λ. Assume that they define foliations in class $\mathcal{C}$ with the same multiplicity m, $m \geqslant 2$, and that*

(i) *the equations $\omega_1 = 0$ and $\omega_2 = 0$ have the same separatrices,*
(ii) $i_j^{(1)} = i_j^{(2)}$, $j = 1, \ldots, m+1$,
(iii) $(h_1^{(1)}, \ldots, h_m^{(1)})$ *and* $(h_1^{(2)}, \ldots, h_m^{(2)})$ *are holomorphically conjugated.*

Then the equations $\omega_1 = 0$ and $\omega_2 = 0$ are holomorphically conjugated.

We close this paper by showing that the method used in Section 3 is not sufficient to cover the remaining cases of singularities in $\Sigma_{n,p} \setminus \Sigma_{n,p}^c$. Let us consider the desingularization of $\Omega = y dy + \cdots = 0$, where Ω belongs to $\Sigma_{n,p}$ as described in Section 1. Now the singularity m_i has only one analytic separatrix, contained in the projective line P_p. The other singularities m_0 and m_{-i} have smooth separatrices transverse to P_p, but the one associated to m_0 is of course contained in P_{p-1}. A *Hopf fibration* for $\Omega = 0$ will be a (*not unique*) linearizable singularity $x\,dy - py\,dx + \cdots = 0$ which contains the transverse separatrices associated to m_0 and m_{-i} as its leaves. We remark that this singularity is desingularized simultaneously with $\Omega = 0$, it has $P_1 \cup \cdots \cap P_{p-1}$ as an invariant set and its leaves are transverse to P_p (in Section 3, we considered such fibrations with the extra condition to contain also the center manifold of m_i). The method in Section 3 consisted in extending a conjugacy between holonomy groups in such a way to respect Hopf fibrations. The next proposition shows this cannot be applied directly to singularities in $\Sigma_{n,p} \setminus \Sigma_{n,p}^c$.

PROPOSITION 5.2. – *There exist* 1*-forms* $\Omega^{(j)}$ *in* $\Sigma_{n,p} \setminus \Sigma^c_{n,p}$, $n < 2p$ $(p \geqslant 2)$ *with associated Hopf fibrations* $G^{(j)}$, $j = 1, 2$, *such that*:

(i) *the holonomy groups* $\mathbf{H}_{\Omega^{(1)}}$ *and* $\mathbf{H}_{\Omega^{(2)}}$ *are holomorphically conjugated,*

(ii) *there exists no holomorphic conjugacy between* $\widetilde{\mathcal{F}}^{(1)}$ *and* $\widetilde{\mathcal{F}}^{(2)}$ *which respect the fibers of* $\widetilde{G}^{(1)}$ *and* $\widetilde{G}^{(2)}$ *in a neighborhood of* P_p ($\widetilde{\mathcal{F}}^{(j)}$ *and* $\widetilde{G}^{(j)}$, $j = 1, 2$, *are the desingularized foliations*).

Proof. – We construct the foliations $\widetilde{\mathcal{F}}^{(1)}$ and $\widetilde{\mathcal{F}}^{(2)}$ using the synthesis method of [5]. We choose first the local models for the singularities $m_i^{(1)}$ and $m_i^{(2)}$:

(1) $\omega_i^{(1)} = y^2\,dx - (x + y^2)\,dy = 0$.

(1′) $\omega_i^{(2)} = \Psi^*\omega_i^{(1)}/U = 0$ where

$$\Psi(x, y) = \big(x, y\big(1 + y^3 g(x, y)\big)\big), \quad g \in \mathcal{O}_2, \tag{36}$$

and U is a unit chosen such that $\omega_i^{(2)} = y^2\,dx - (x + y^2 C(x, y))\,dy$ for some C in $\mathcal{O}_2$ with $C(0, 0) \neq 0$.

By definition, $\omega_i^{(1)}$ and $\omega_i^{(2)}$ have their formal separatrices tangent to the y axis.

LEMMA 5.3. – Ψ *is the unique conjugacy between the equations* $\omega_i^{(1)} = 0$ *and* $\omega_i^{(2)} = 0$ *which is fibered in the variable* $x \in \mathbf{C}$.

Proof. – Let Φ be another conjugacy between the equations which is fibered in the variable $x \in \mathbf{C}$; then $\Psi \circ \Phi^{-1}$ satisfies

$$\big(\Psi \circ \Phi^{-1}\big)^*\omega_i^{(1)} \wedge \omega_i^{(1)} = 0. \tag{37}$$

So it is enough to prove that any diffeomorphism F of the form $F(x, y) = (x, yf(x, y))$ that satisfies

$$F^*\omega_i^{(1)} \wedge \omega_i^{(1)} = 0 \tag{38}$$

must be the identity map. A simple computation shows that (38) is equivalent to

$$\begin{aligned} &y^3 f^2\big(-1 + f + y(f'_x + f'_y)\big) \\ &\quad + xy^2\big(f'_y + f'_x + f^2 f'_x\big) + xyf(1 - f) + x^2 f'_x = 0, \end{aligned} \tag{39}$$

where f'_x (respectively f'_y) denotes the partial derivative of f with respect to x (respectively y). The relation (39) shows that $f(0,0)=1$ and that y divises f'_x so that $f(x,y)=1+y\alpha_1(x,y)$. We proceed now by induction.

– Step (1) of the induction: $f=1+y\alpha_1,\ \alpha_1\in\mathcal{O}_2$.
– Hypothesis of induction: $f=1+y^p\alpha_p,\ \alpha_p\in\mathcal{O}_2$.

We are going to prove that $f=1+y^{p+1}\alpha_{p+1}$ where α_{p+1} is an element of $\mathcal{O}_2$. From (39) we deduce that y^{p+1} divises f'_x so that

$$f(x,y)=1+y^{p+1}\alpha_{p+1}(x,y)+P_p(y), \tag{40}$$

where P_p is a polynomial of degree less or equal to p. The induction hypothesis implies that $P(y)=by^p$ where b is a complex number. We have then

$$f(x,y)=1+y^{p+1}\alpha_{p+1}(x,y)+by^p \tag{41}$$

and

$$f'_x(x,y)=y^{p+1}(\alpha_{p+1})'_x(x,y), \tag{42}$$

$$f'_y(x,y)=(p+1)y^p\alpha_{p+1}(x,y)+y^{p+1}(\alpha_{p+1})'_y(x,y)+pby^{p-1}.$$

Using again (39) we see that x divises $f-1+y(f'_x+f'_y)$ so that

$$\begin{aligned} &y^{p+1}\alpha_{p+1}+by^p+y^{p+2}(\alpha_{p+1})'_x+(p+1)y^{p+1}\alpha_{p+1}\\ &\quad+y^{p+2}(\alpha_{p+1})'_y+pby^p=x\delta \end{aligned} \tag{43}$$

for some δ in $\mathcal{O}_2$. Therefore $b=0$ and $f=1+y^n\alpha_n,\ \alpha_n\in\mathcal{O}_2$, for all $n\geqslant 1$. $\square$

We go on constructing the foliations $\widetilde{\mathcal{F}}^{(1)}$ and $\widetilde{\mathcal{F}}^{(2)}$. We complete $\widetilde{\mathcal{F}}^{(1)}$ in the following way:

(2) At $m_0^{(1)}$ the local model is $\omega_0^{(1)}=x\,dy+\frac{1-1}{p}y\,dx=0$;

(3) At $m_{-i}^{(1)}$ the local model is a resonnant singularity $x\,dy+\frac{1}{p}y(1+\cdots)\,dx$ such that the relation $h_i^{(1)}\circ h_0^{(1)}=(h_{-i}^{(1)})^{-1}$ between the local holonomies of $\{y=0\}$ holds;

(4) The desingularization scheme is the one expected for singularities in $\Sigma_{n,p}$.

According to [5] we can glue these local models to obtain $\widetilde{\mathcal{F}}^{(1)}$ which corresponds to an element $\Omega^{(1)}$ in $\Sigma_{n,p}\setminus\Sigma_{n,p}^c$, $n>2p$ (proceed as in [9]) and whose holonomy group $\mathbf{H}_{\Omega^{(1)}}$ is conjugated to $(h_0^{(1)}, h_i^{(1)})$. Since the gluing diffeomorphisms respect the local fibration $dx = 0$, we get a fibration transverse to D_p which contains the separatrices of $m_{-i}^{(1)}$ and $m_0^{(1)}$. This last condition allows us (still using [5]) to complete the fibration to a Hopf fibration $\widetilde{G}^{(1)}$ associated to $\widetilde{\mathcal{F}}^{(1)}$.

To construct $\widetilde{\mathcal{F}}^{(2)}$ we have to add to the local model (1′) at $m_i^{(2)}$ the ones at $m_0^{(2)}$ and $m_{-i}^{(2)}$. Let us consider a germ φ of $\mathrm{Diff}(\mathbf{C}, 0)$ defined in some section $\Sigma = \{x = x_0\}$ close to $0 \in \mathbf{C}$ which conjugates $h_i^{(1)}$ and $h_i^{(2)}$: $h_i^{(2)} = \varphi^* h_i^{(1)}$ (later we will specify φ). The local models are given by:

(2′) At $m_0^{(2)}$: $\omega_0^{(2)} = x\,dy + \frac{1-1}{p} y(1 + \cdots)\,dx = 0$ such that the holonomy diffeomorphism $h_0^{(2)}$ satisfies $h_0^{(2)} = \varphi^* h_0^{(1)}$.

(3′) At $m_{-i}^{(2)}$: $\omega_{-i}^{(2)} = x\,dy + \frac{1}{p} y(1 + \cdots)\,dx = 0$ with $h_{-i}^{(2)} = \varphi^* h_{-i}^{(1)}$.

(4′) Analogous to (4).

Since $h_i^{(2)} \circ h_0^{(2)} = (h_{-i}^{(2)})^{-1}$, by [5] and [9] there exist $\widetilde{\mathcal{F}}^{(2)}$ corresponding to $\Omega^{(2)}$ in $\Sigma_{n,p} \setminus \Sigma_{n,p}^c$, $n > 2p$, whose holonomy group $\mathbf{H}_{\Omega^{(2)}}$ is conjugated to $(h_0^{(2)}, h_i^{(2)})$ and an associated Hopf fibration $\widetilde{G}^{(2)}$.

Suppose now that there exists some holomorphic Φ fibered with respect to the Hopf fibrations $\widetilde{G}^{(1)}$ and $\widetilde{G}^{(2)}$ which conjugates $\widetilde{\mathcal{F}}^{(1)}$ and $\widetilde{\mathcal{F}}^{(2)}$. Let us denote $\xi_i^{(1)}$ (respectively $\xi_i^{(2)}$) an embedding which takes model (1) (respectively (1′)) into $\widetilde{\mathcal{F}}^{(1)}$ (respectively $\widetilde{\mathcal{F}}^{(2)}$) in a neighborhood of $m_i^{(1)}$ (respectively $m_i^{(2)}$) and which also takes the fibration $dx = 0$ into $\widetilde{G}^{(1)}$. The construction in [5] allows us to suppose that $(\xi_i^{(1)})^* \mathbf{H}_{\Omega^{(1)}} = (h_0^{(1)}, h_i^{(1)})$ and $(\xi_i^{(2)})^* \mathbf{H}_{\Omega^{(2)}} = (h_0^{(2)}, h_i^{(2)})$ in the section Σ. Therefore in Σ:

$$\left(\left(\xi_i^{(1)}\right)^{-1} \circ \Phi \circ \xi_i^{(2)}\right)^* \left(h_0^{(1)}, h_i^{(1)}\right) = \left(h_0^{(2)}, h_i^{(2)}\right). \tag{44}$$

From Lemma 5.3 we get then

$$\Psi_0^* \left(h_0^{(1)}, h_i^{(1)}\right) = \left(h_0^{(2)}, h_i^{(2)}\right), \tag{45}$$

where $\Psi_0 = \Psi_{|\Sigma}$. More precisely:

$$h_0^{(1)} \circ \Psi_0 = \Psi_0 \circ h_0^{(2)}. \tag{46}$$

Using the relation $h_0^{(1)} \circ \varphi = \varphi \circ h_0^{(2)}$, we obtain that

$$h_0^{(1)} \circ \varphi \circ \Psi_0^{-1} = \varphi \circ \Psi_0^{-1} \circ h_0^{(1)}. \tag{47}$$

Let us put $\varphi = h_i^{(1)} \circ \Psi_0$, which is possible since $\varphi \circ h_i^{(2)} = h_i^{(1)} \circ \Psi_0 \circ h_i^{(2)} = h_i^{(1)} \circ h_i^{(1)} \circ \Psi_0 = h_i^{(1)} \circ \varphi$. Therefore

$$h_0^{(1)} \circ h_i^{(1)} = h_i^{(1)} \circ h_0^{(1)}. \tag{48}$$

This equality does not hold since $h_i^{(1)}$ has a dynamic with only two petals so that the diffeomorphisms that commute with it are tangent to the identity map; this is not the case for $h_0^{(1)}$. $\square$

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